MTH 201
Multivariable calculus and differential equations
Homework 9
Green's theorem, Stokes' theorem, and Divergence theorem

Green's theorem

1. Use Green's theorem to evaluate the line integral along the given positively oriented curve
   (a) $\oint_C xydy - y^2dx$, where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.
   (b) $\oint_C xydx + x^2y^3dy$, where C is the triangular curve with vertices $(0,0), (1,0)$, and $(1,2)$ with positive orientation.
   (c) $\oint_C y^2dy + 3xydx$, where C is the positively oriented boundary of semi-annular region in the upper half plane between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

2. Evaluate the line integral $\oint_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy$, where C is the positively oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

3. Verify Green's theorem on the annular region $D : 0.5 \leq x^2 + y^2 \leq 1$ for the vector field $\mathbf{F}(x, y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$.

4. Evaluate the line integral $\oint_C \mathbf{F} \cdot dr$, where $\mathbf{F}(x, y) = \frac{2xy}{(x^2+y^2)^2}\vec{i} + \frac{y^2-x^2}{(x^2+y^2)^2}\vec{j}$ and C is a positively oriented simple closed curve that encloses the origin.

5. Let $\mathbf{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ be a vector field on an open simply connected domain D such that P and Q have continuous first order partial derivatives and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on D. Use Green's theorem to show that $\mathbf{F}$ is a conservative vector field.

Stokes' theorem

6. Use Stokes' Theorem to compute the surface integral $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where
   (a) $\mathbf{F}(x, y, z) = 2y\cos z\vec{i} + e^x \sin z\vec{j} + xe^y\vec{k}$ and the surface S is the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$ oriented upward.
   (b) $\mathbf{F}(x, y, z) = z^2\vec{i} - 3xy\vec{j} + x^3y^3\vec{k}$ and the surface S is the part of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$ with upward orientation.

7. Use Stokes' theorem to evaluate the line integral $\int_C \mathbf{F} \cdot dr$, where $\mathbf{F}(x, y, z) = z^2\vec{i} + y^2\vec{j} + x\vec{k}$ and C is the triangle with vertices $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ with counter-clockwise rotation.

8. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral $\int_C zdx - 2xdy + 3ydz$ depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

Divergence theorem

9. Verify that the divergence theorem is true for the vector field $\mathbf{F}$ over the solid region R

(a) $\mathbf{F}(x,y,z) = 3x\overrightarrow{i} + xy\overrightarrow{j} + 2xz\overrightarrow{k}$ and R is the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$, and $z = 1$.

(b) $\mathbf{F}(x,y,z) = x^2\overrightarrow{i} + xy\overrightarrow{j} + z\overrightarrow{k}$ and R is the solid region bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy-$plane.

10. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = z\overrightarrow{i} + y\overrightarrow{j} + x\overrightarrow{k}$ and S is the boundary surface of the solid ball $x^2 + y^2 + z^2 \leq 16$.

11. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = xy\overrightarrow{i} - \frac{y^2}{2}\overrightarrow{j} + z\overrightarrow{k}$ and the surface S consists of the three surfaces $z = 4 - 3x^2 - 3y^2,\ 1 \leq z \leq 4$ on the top, $x^2 + y^2 = 1,\ 0 \leq z \leq 1$ on the sides, and $z = 0$ on the bottom.